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Generalized KKM type theorems in FC-spaces with applications (II)

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Abstract This paper is a continuum of the preceding paper of author. By applying a coincidence theorem in noncompact FC-space without any convexity structure due to author, a new KKM type theorem is first proved under noncompact setting of FC-spaces. The equivalent relation between the coincidence theorem and the KKM type theorem is also established. As applications of the KKM type theorem, we establish some new existence theorems of solutions for three classes of generalized vector equilibrium problems under noncompact setting of FC-spaces. These theorems improve and generalize many known results in literature.

Keywords Coincidence theorem · KKM type theorem · Generalized vector equilibrium problem · Transfer compactly open-valued (closed-valued) mapping · C-diagonally quasi-convex · Proper quasimonotone · FC-space

1 Introduction

In 1980, Giannessi [16] first introduced the vector variational inequality problem in finite dimensional Euclidean spaces. Since then, such problem has been extended and generalized by many authors in various different directions. Motivation for this comes from the fact that vector variational inequality and its various generalizations have extensive and important applications in vector optimization, optimal control, mathematical programing, operations research, and equilibrium problem of economics. Inspired and motivated by above applications, various generalized vector variational inequality problems, generalized vector equilibrium problems have become important developed directions of vector variational inequality theory, (for example, see [2, 5–7, 9, 14, 15, 17, 19, 21–27]).

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Let X, Y be topological spaces, Z be a nonempty set and 2^Z be the family of all subsets of Z. Let $F: X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings.

A generalized vector equilibrium problem of type (I) (GVEP(I)) is to find $\hat{x} \in X$ such that

$$F(\hat{x}, y) \subset C(\hat{x}), \quad \forall\, y \in Y.$$

A generalized vector equilibrium problem of type (II) (GVEP(II)) is to find $\hat{x} \in X$ such that

$$F(\hat{x}, y) \not\subset C(\hat{x}), \quad \forall\, y \in Y.$$

A generalized vector equilibrium problem of type (III) (GVEP(III)) is to find $\hat{x} \in X$ such that

$$F(\hat{x}, y) \bigcap C(\hat{x}) \neq \emptyset, \quad \forall\, y \in Y.$$

For appropriate choices of the spaces X, Y, Z, and the mappings F and C, it is easy to see that the GVEP(I) – GVEP(III) include most extensions and generalizations of generalized vector equilibrium problems and generalized vector variational inequality problems as very special cases (for example, see [2, 5–7, 9, 14, 15, 17, 19, 21–27] and the references therein).

Recently Lin et al. [21], Lin and Chen [22], and Lin and Wang [23] further study the class KKM(X, Y) of set-valued mappings with KKM property in topological vector spaces. They established some new KKM type theorems, coincidence theorems, and the equivalent relations between the KKM type theorems and coincidence theorems. As applications, some existence theorems of solutions for the GVEP(I) and GVEP(III) are also proved under suitable assumptions. By using fixed point theorem and generalized KKM theorem, Ding and Park [14, 15] proved some existence theorems of solutions for the GVEP(II) in G-convex spaces under different assumptions.

In most of known KKM type theorems and coincidence theorems, the convexity assumptions play a crucial role which strictly restricts the applicable area of these KKM type theorems and coincidence theorems. In [13], we have generalized the KKM type theorems and coincidence theorems of [21] from topological vector spaces to FC-spaces without any convexity structure under much weak assumptions.

In this paper, by applying our coincidence theorem in [13], an KKM type theorem is first proved under noncompact setting of FC-spaces without any convexity structure. The equivalent relation between the coincidence theorem and the KKM type theorem is also established. As applications of the KKM type theorem, we establish some new existence theorems of solutions for the GVEP(I)–GVEP(III) under noncompact setting of FC-spaces. These theorems improve and generalize many known results in literature.

2 Preliminaries

Let Δ_n be the standard n-dimensional simplex with vertices $e_0, e_1, \ldots, e_n$. If J is a nonempty subset of $\{0, 1, \ldots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$. For topological space X, a subset A of X is said to be compactly open (resp., compactly closed) if for each nonempty compact subset K of X, $A \cap K$ is open (resp., closed) in K. The compact closure and the compact interior of A (see [10]) are defined by

$$\mathrm{ccl}(A) = \bigcap\{B \subset X\colon A \subset B \text{ and } B \text{ is compactly closed in } X\},$$
$$\mathrm{cint}(A) = \bigcup\{B \subset X\colon B \subset A \text{ and } B \text{ is compactly open in } X\}.$$

It is easy to see that for each nonempty compact subset K of X, we have $\mathrm{ccl}(A)\cap K = \mathrm{cl}_K(A\cap K)$, $\mathrm{cint}(A)\cap K = \mathrm{int}_K(A\cap K)$ and $\mathrm{ccl}(X\setminus A) = X\setminus\mathrm{cint}(A)$. A subset A of X is compactly open (resp., compactly closed) if and only if $\mathrm{cint}(A) = A$ (resp., $\mathrm{ccl}(A) = A$).

Let X and Y be topological spaces. A set-valued mapping $T: X \to 2^Y$ is said to be transfer compactly open-valued (resp., transfer compactly closed-valued) on X (see [10]) if for each $x \in X$, each nonempty compact subset K of Y and each $y \in K$, $y \in T(x)\cap K$ (resp., $y \notin T(x)\cap K$) implies that there exists $x' \in X$ such that $y \in \mathrm{inf}_K(T(x')\cap K)$ (resp., $y \notin \mathrm{cl}_K(T(x')\cap K)$). We observe that the notion of transfer compact-open (resp. transfer compact-closed) mappings defined by Lin and Ansari in [20, p 409] is a special cases of the above corresponding notion.

The following notion was introduced by Ding [12].

Definition 2.1 (Y,φ_N) is said to be a finitely continuous space (in short, FC-space) if Y is a topological space and for each $N = \{y_0,\ldots,y_n\} \in \langle Y\rangle$ where some elements may be same, there exists a continuous mapping $\varphi_N\colon \Delta_n \to Y$. If A and B are two subsets of Y, B is said to be a FC-subspace of Y relative to A if for each $N = \{y_0,\ldots,y_n\} \in \langle Y\rangle$ and for any $\{y_{i_0}\ldots,y_{i_k}\} \subset A\cap\{y_0,\ldots,y_n\}$, $\varphi_N(\Delta_k) \subset B$ where $\Delta_k = \mathrm{co}(\{e_{i_0},\ldots,e_{i_k}\})$. If $A = B$, then B is called a FC-subspace of Y.

It is easy to see that the class of FC-spaces includes the classes of convex sets in topological vector spaces, C-spaces (or H-spaces) [18], G-convex spaces [28], L-convex spaces [4], and many topological spaces with abstract convexity structure as true subclasses. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces.

Definition 2.2 Let (Y,φ_N) be a FC-space and X be a topological space. Let $T, F: Y \to 2^X$ be two set-valued mappings. F is said to be a generalized KKM mapping with respect to T if for each $N = \{y_0,\ldots,y_n\} \in \langle Y\rangle$ and each $\{y_{i_0},\ldots,y_{i_k}\} \subset N$, $T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} F(y_{i_j})$ where $\Delta_k = \mathrm{co}(\{e_{i_0},\ldots,e_{i_k}\})$. T is said to have the KKM property if for each generalized KKM mapping F with respect to T, the family $\{\overline{F(y)}\colon y \in Y\}$ has the finite intersection property. Write

$$\mathrm{KKM}(Y,X) = \{T\colon Y \to 2^X\colon T \text{ has the KKM property}\}.$$

Clearly, the new class $\mathrm{KKM}(Y,X)$ generalizes the classes $\mathrm{KKM}(Y,X)$ in [8, 21] from convex subsets of topological vector spaces to FC-spaces.

Lemma 2.1 [1] *Let X and Y be topological spaces and $G\colon X \to 2^Y$ be a set-valued mapping. Then G is lower semicontinuous in $x \in X$ if and only if for any $y \in G(x)$ and any net $\{x_\alpha\} \subset X$ satisfying $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in G(x_\alpha)$ and $y_\alpha \to y$.*

3 Some equivalent coincidence theorems and KKM type theorems

The following result is Theorem 4.4 of our preceding paper [13].

Theorem 3.1 *Let* $(Y, \{\varphi_N\})$ *be a* FC*-space and* X *be a topological space. Let* $T \in \mathrm{KKM}(Y, X)$ *and* $H, P, Q: X \to 2^Y$ *be set-valued mappings such that*

(1) Q^{-1} *is transfer compactly open-valued and for each* $x \in X$, $Q(x) \neq \emptyset$,
(2) *for each* $x \in X$, $Q(x) \subset H(x)$,
(3) *for each* $x \in X$, $P(x)$ *is a* FC*-subspace of* Y *relative to* $H(x)$.
(4) *for each compact subset* D *of* Y, $\overline{T(D)}$ *is compact in* X,
(5) *there exist a compact subset* K *of* X *such that for each* $N \in \langle Y\rangle$, *there is a compact* FC*-subspace* L_N *of* Y *containing* N *satisfying*

$$T(L_N)\backslash K \subset \bigcup_{y \in L_N} \mathrm{cint} Q^{-1}(y).$$

Then there exists $(y_0, x_0) \in Y \times X$ such that $x_0 \in T(y_0)$ and $y_0 \in P(x_0)$.

Corollary 3.1 *Let* $(Y, \{\varphi_N\})$ *be a* FC*-space and* X *be a topological space. Let* $T \in \mathrm{KKM}(Y, X)$ *and* $Q: X \to 2^Y$ *be set-valued mappings such that*

(1) Q^{-1} *is transfer compactly open-valued and for each* $x \in X$, $Q(x) \neq \emptyset$,
(2) *for each* $x \in X$, $Q(x)$ *is a* FC*-subspace of* Y.
(3) *for each compact subset* D *of* Y, $\overline{T(D)}$ *is compact in* X,
(4) *there exist a compact subset* K *of* X *such that for each* $N \in \langle Y\rangle$, *there is a compact* FC*-subspace* L_N *of* Y *containing* N *satisfying*

$$T(L_N)\backslash K \subset \bigcup_{y \in L_N} \mathrm{cint} Q^{-1}(y).$$

Then there exists $(y_0, x_0) \in Y \times X$ such that $x_0 \in T(y_0)$ and $y_0 \in Q(x_0)$.

Proof The conclusion of Corollary 3.1 holds from Theorem 3.1 with $Q = H = P$.

Remark 3.1 From Lemma 2.1 of Ding [13] it is easy to see that Theorem 3.1 and Corollary 3.1 generalize Theorem 2.6 of Lin et al. [21], Theorems 3.1 and 3.2 of Lin and Wan [23] and Theorems 3.5 and 3.7 of Lin and Chen [22] from topological vector spaces to FC-spaces without any convexity structure under weaker assumptions. Theorem 3.1 also generalize Theorem 3.4 of Ding [11] from the class $\mathcal{U}_c^k(Y, X)$ to the class $\mathrm{KKM}(Y, X)$ and from L-convex spaces to FC-spaces.

The following KKM type theorem is an equivalent statement of Theorem 3.1.

Theorem 3.2 *Let* $(Y, \{\varphi_N\})$ *be a* FC*-space and* X *be a topological space. Let* $T \in \mathrm{KKM}(Y, X)$ *and* $F, G, M: Y \to 2^X$ *be set-valued mappings such that*

(1) *F has transfer compactly closed values,*
(2) *for each* $y \in Y$, $G(y) \subset F(y)$ *and* $T(y) \subset M(y)$,
(3) *G is a generalized* KKM *mapping with respect to* M.
(4) *for each compact subset* D *of* Y, $\overline{T(D)}$ *is compact in* X,

(5) *there exists a compact subset K of X such that, for each $N \in \langle Y\rangle$, there exists a compact* FC-*subspace L_N of Y containing N such that*

$$T(L_N)\bigcap\Big(\bigcap_{y\in L_N}\mathrm{ccl}F(y)\Big)\subset K.$$

Then $\bigcap_{y\in Y}F(y)\neq\emptyset$.

Proof Theorem 3.1 $\Rightarrow$ Theorem 3.2. Suppose the conclusion of Theorem 3.2 is false, then $\cap_{y\in Y}F(y)=\emptyset$. Hence for each $x\in X$, $Y\backslash F^{-1}(x)\neq\emptyset$. Define set-valued mappings $Q,P,H\colon X\to 2^Y$ by

$$Q(x)=Y\backslash F^{-1}(x),\qquad H(x)=Y\backslash G^{-1}(x)\quad\text{and}\quad P(x)=Y\backslash M^{-1}(x),\quad\forall\ x\in X.$$

Then for each $x\in X$, $Q(x)\neq\emptyset$, and for each $y\in Y$,

$$Q^{-1}(y)=\{x\in X\colon y\in Q(x)\}=\{x\in X\colon y\notin F^{-1}(x)\}=\{x\in X\colon x\notin F(y)\}=X\backslash F(y).$$

By (1) and Lemma 2.2 in [13], $Q^{-1}\colon Y\to 2^X$ is transfer compactly open-valued. The condition (1) of Theorem 3.1 is satisfied. Since $G(y)\subset F(y)$ for each $y\in Y$ by (2), we have that $Q(x)\subset H(x)$ for each $x\in X$. The condition (2) of Theorem 3.1 is satisfied. By (3) and Lemma 2.5 in [13], for each $x\in X$, $P(x)$ is a FC-subspace of Y with respect to $H(x)$. The condition (3) of Theorem 3.1 is satisfied. Since $Q^{-1}(y)=X\backslash F(y)$ for each $y\in Y$, by (v), we have

$$T(L_N)\backslash K\subset\bigcup_{y\in L_N}(X\backslash\mathrm{ccl}F(y))\subset\bigcup_{y\in L_N}\mathrm{cint}(X\backslash F(y))=\bigcup_{y\in L_N}\mathrm{cint}Q^{-1}(y).$$

All conditions of Theorem 3.1 are satisfied. By Theorem 3.1 there exists $(y_0,x_0)\in Y\times X$ such that $x_0\in T(y_0)$ and $y_0\in P(x_0)$. Hence we have $y_0\notin M^{-1}(x_0)$ and so $x_0\notin M(y_0)$. By (2), $x_0\notin T(y_0)$ which is a contradiction. Hence $\cap_{y\in Y}F(y)\neq\emptyset$.

Theorem 3.2 $\Rightarrow$ Theorem 3.1. Under the assumptions of Theorem 3.1, if the conclusion of Theorem 3.1 is false, then for each $y\in Y$, $T(y)\cap P^{-1}(y)=\emptyset$ which implies $T(y)\subset X\backslash P^{-1}(y)$. Define set-valued mappings $F,G,M\colon Y\to 2^X$ by

$$F(y)=X\backslash Q^{-1}(y),\qquad G(y)=X\backslash H^{-1}(y)\quad\text{and}\quad M(y)=X\backslash P^{-1}(y),\quad\forall\ y\in Y.$$

Hence $T(y)\subset M(y)$ for each $y\in Y$. By the condition (1) of Theorem 3.1 and Lemma 2.2 in [13], F has transfer compactly closed values. By condition (2) of Theorem 3.1 and the definition of F and G, we have $G(y)\subset F(y)$ for each $y\in Y$. By (3) and Lemma 2.5 in [13], G is a generalized KKM mapping with respect to M. By condition (v) of Theorem 3.1, we have

$$T(L_N)\backslash\bigcup_{y\in L_N}\mathrm{cint}Q^{-1}(y)\subset K.$$

It follows that

$$T(L_N)\bigcap\Big(\bigcap_{y\in L_N}\mathrm{ccl}F(y)\Big)=T(L_N)\bigcap\Big(\bigcap_{y\in L_N}\mathrm{ccl}(X\backslash Q^{-1}(y))=T(L_N)\backslash\bigcup_{y\in L_N}\mathrm{cint}Q^{-1}(y)\Big)\subset K.$$

All conditions of Theorem 3.2 are satisfied. By Theorem 3.2, $\cap_{y\in Y}F(y)\neq\emptyset$. Any take $x_0\in\cap_{y\in Y}F(y)$, then $x_0\in F(y)=X\backslash Q^{-1}(y)$ for all $y\in Y$. Therefore $y\notin Q(x_0)$ for all $y\in Y$ and so $Q(x_0)=\emptyset$ which contradicts the condition (1) of Theorem 3.1. Hence the conclusion of Theorem 3.1 holds.

The following KKM type theorem is an equivalent statement of Corollary 3.1.

Corollary 3.2 *Let* $(Y, \{\varphi_N\})$ *be a* FC-*space and* X *be a topological space. Let* $T \in \mathrm{KKM}(Y, X)$ *and* $F\colon Y \to 2^X$ *be set-valued mappings such that*

(1) *F has transfer compactly closed values,*
(2) *F is a generalized* KKM *mapping with respect to T.*
(3) *for each compact subset D of Y,* $\overline{T(D)}$ *is compact in X,*
(4) *there exists a compact subset K of X such that, for each* $N \in \langle Y \rangle$*, there exists a compact* FC-*subspace* L_N *of Y containing N such that*

$$T(L_N) \bigcap (\bigcap_{y \in L_N} \mathrm{ccl}F(y) \subset K.$$

Then $\cap_{y \in Y} F(y) \neq \emptyset$.

Proof The conclusion of Corollary 3.2 holds from Theorem 3.2 with $G = F$ and $M = T$

Remark 3.2 Theorem 3.2 and Corollary 3.2 improve and generalize Theorem 3.3 of Lin and Wan [23] and Theorem 3.2 of Lin and Chen [22] from topological vector spaces to FC-spaces without any convexity structure under much weak assumptions. Theorem 3.2 and Corollary 3.2 also generalize Theorems 3.2 and 3.3 of Ding [11] from the class $\mathcal{U}_c^k(Y, X)$ to the class KKM(Y, X) and from L-convex spaces to FC-spaces.

4 Existence results of generalized vector equilibrium problems

The following notion of properly quasimonotone bimapping was introduced by Bianchi and Pini [6] in topological vector spaces.

Definition 4.1 Let $(Y, \{\varphi_N\})$ be a FC-space, X be a topological space and Z be a nonempty sets. Let $T\colon Y \to 2^X$, $F\colon X \times Y \to 2^Z$ and $C\colon X \to 2^Z$ be set-valued mappings. $F(x, y)$ is said to be a properly quasimonotone mapping of Type (I) (resp., Types (II) and (III)) with respect to T on $X \times Y$ if the mapping $F^*\colon Y \to 2^X$ defined by

$$F^*(y) = \{x \in X\colon F(x, y) \subset C(x))\}$$
$$(\text{resp., } F^*(y) = \{x \in X\colon F(x, y) \not\subset C(x)\},\ F^*(y) = \{x \in X\colon F(x, y) \cap C(x)) \neq \emptyset\}s)$$

is a generalized KKM mapping with respect to T.

Definition 4.2 Let $(Y, \{\varphi_N\})$ be a FC-space, X be a topological space and Z be a nonempty set. Let $T\colon Y \to 2^X$, $F\colon X \times Y \to 2^Z$ and $C\colon X \to 2^Z$ be set-valued mappings. $F(x, y)$ is said to be a C-diagonally quasiconvex mapping of Type (I) (resp., Type (II), Type (III)) with respect to T in second argument if, for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N$, and each $x \in T(\varphi_N(\Delta_N))$, there exists $j \in \{0, \cdots, k\}$ such that

$$F(x, y_{i_j}) \subset C(x) \ \ (\text{resp., } F(x, y_{i_j}) \not\subset C(x),\ \ F(x, y_{i_j}) \bigcap C(x) \neq \emptyset).$$

If $X = Y$ and T is the identity mapping, then $F(x, y)$ is said to be a C-diagonally quasiconvex mapping of Type (I) (resp., Type (II), Type (III)) in second argument.

Remark 4.1 Definition 4.2 extend the corresponding notions of Hou et al. [19] from topological vector spaces to FC-spaces under more general setting.

Definition 4.3 Let X and Y be topological spaces and Z be nonempty sets. Let $F: X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings. $F(x,y)$ is said to be a C-transfer compactly continuous mapping of Type (I) (resp., Type (II), Type (III)) in first argument if, for any compact subset K of X and any $x \in K$, $\{y \in Y: F(x,y) \not\subset C(x)\} \neq \emptyset$ (resp., $\{y \in Y: F(x,y) \subset C(x)\} \neq \emptyset$, $\{y \in Y: F(x,y) \cap C(x) = \emptyset\} \neq \emptyset$) implies that there exist a relatively open neighborhoos $N(x)$ of x in K and a point $y' \in Y$ such that $F(z,y') \not\subset C(z)$ (resp., $F(z,y') \subset C(z)$, $F(z,y') \cap C(z) = \emptyset$) for all $z \in N(x)$.

Remark 4.2 Definition 4.3 extend the corresponding notion of Ding and Park [15].

Proposition 4.1 *Let* $(Y, \{\varphi_N\})$ *be a* FC*-space,* X *be a topological space and* Z *be a nonempty set. Let* $T: Y \to 2^X$, $F: X \times Y \to 2^Z$, *and* $C: X \to 2^Z$ *be set-valued mappings.*

Then $F(x,y)$ is a properly quasimonotone mapping of Type (I) (resp., Types (II) and Type (III)) with respect to T if and only if $F(x,y)$ is a C-diagonally quasiconvex mapping of Type (I) (resp., Types (II)and (III)) with respect to T in second argument.

Proof Suppose that $F(x,y)$ is a properly quasimonotone mapping of Type (I) (resp., Types (II) and (III)) with respect to T, then for each $N = \{y_0, \dots, y_n\} \in< Y >$, each $\{y_{i_0}, \dots, y_{i_k}\} \subset N$,

$$T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} \{x \in X: F(x, y_{i_j}) \subset C(x)\} \tag{4.1}$$

$$(\text{resp., } T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} \{x \in X: F(x, y_{i_j}) \not\subset C(x)\}, \tag{4.2}$$

$$T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} \left\{x \in X: F(x, y_{i_j}) \bigcap C(x) \neq \emptyset\right\}. \tag{4.3}$$

If $F(x,y)$ is not a C-diagonally quasiconvex mapping of Type (I) (resp., Types (II) and (III)) with respect to T in second argument, then there exist $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$, $\{y_{i_0}, \dots, y_{i_k}\} \subset N$ and $x_0 \in T(\varphi_N(\Delta_k)$ such that

$$F(x_0, y_{i_j}) \not\subset C(x_0) \;\; (\text{resp., } F(x_0, y_{i_j}) \subset C(x_0),\; F(x_0, y_{i_j}) \bigcap C(x_0) = \emptyset\})$$

for all $j = 0, \dots, k$. It follows that $x_0 \in T(\varphi_N(\Delta_k))$ and $x_0 \notin \cup_{j=0}^{k} \{x \in X: F(x, y_{i_j}) \subset C(x)\}$ (resp., $x_0 \notin \cup_{j=0}^{k} \{x \in X: F(x, y_{i_j}) \subset C(x)\}$, $x_0 \notin \cup_{j=0}^{k} \{x \in X: F(x, y_{i_j}) \cap C(x) \neq \emptyset\}$) which contradicts (4.1) (resp., (4.2), (4.3)). Now suppose that $F(x,y)$ is a C-diagonally quasiconvex mapping of Type (I) (resp., Type (II), Type (III)) with respect to T in second argument, then for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$, each $\{y_{i_0}, \dots, y_{i_k}\} \subset N$, and each $x \in T(\varphi_N(\Delta_N))$, there exists $j \in \{0, \dots, k\}$ such that

$$F(x, y_{i_j}) \subset C(x) \tag{4.4}$$

$$(\text{resp., } F(x, y_{i_j}) \not\subset C(x), \tag{4.5}$$

$$F(x, y_{i_j}) \bigcap C(x) \neq \emptyset. \tag{4.6}$$

If $F(x,y)$ is not a properly quasimonotone mapping of Type (I) (resp., Types (II) and (III)) with respect to T, then there exist $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$, $\{y_{i_0}, \dots, y_{i_k}\} \subset N$ such that $T(\varphi_N(\Delta_k)) \not\subset \bigcup_{j=0}^{k}\{x \in X : F(x, y_{i_j}) \subset C(x)\}$ (resp., $T(\varphi_N(\Delta_k)) \not\subset \bigcup_{j=0}^{k}\{x \in X : F(x, y_{i_j}) \not\subset C(x)\}$, $T(\varphi_N(\Delta_k)) \not\subset \bigcup_{j=0}^{k}\{x \in X : F(x, y_{i_j}) \cap C(x) \neq \emptyset\}$). Hence there exists $x_0 \in T(\varphi_N(\Delta_k))$ such that

$$F(x_0, y_{i_j}) \not\subset C(x_0) \ \ (\text{resp., } F(x_0, y_{i_j}) \subset C(x_0),\ F(x_0, y_{i_j}) \bigcap C(x_0) = \emptyset\}$$

for all $j = 0, \dots, k$ which contradicts (4.4) (resp., (4.5), (4.6)). This completes the proof.

Proposition 4.2 *Let $(Y, \{\varphi_N\})$ be a* FC*-space, X be a topological space and Z be a nonempty set. Let $T : Y \to 2^X$, $F : X \times Y \to 2^Z$, and $C : X \to 2^Z$ be set-valued mappings. Suppose that for each $x \in X$, $Y \setminus T^{-1}(x)$ is a* FC*-subspace of Y relative to $\{y \in Y : F(x,y) \not\subset C(x)\}$ (resp., $\{y \in Y : F(x,y) \subset C(x)\}$, $\{y \in Y : F(x,y) \cap C(x) = \emptyset\}$). Then $F(x,y)$ is a properly quasimonotone mapping of Type* (I) (*resp., Types* (II) *and* (III)) *with respect to T.*

Proof Define a set-valued mapping $F^* : Y \to 2^X$ by

$$F^*(y) = \{x \in X : F(x,y) \subset C(x)\}$$
$$(\text{resp., } F^*(y) = \{x \in X : F(x,y) \not\subset C(x)\},\ F^*(y) = \{x \in X : F(x,y) \bigcap C(x) \neq \emptyset\}).$$

Then for each $x \in X$, we have

$$Y \setminus (F^*)^{-1}(x) = \{y \in Y : F(x,y) \not\subset C(x)\}$$
$$(\text{resp., } Y \setminus (F^*)^{-1}(x) = \{y \in Y : F(x,y) \subset C(x)\},\ Y \setminus (F^*)^{-1}(x) = \{y \in Y : F(x,y) \cap C(x) = \emptyset\}).$$

By the assumption, for each $x \in X$, $Y \setminus T^{-1}(x)$ is a FC-subspace of Y relative to $Y \setminus (F^*)^{-1}(x)$. By Lemma 2.5 of Ding [13], F^* is a generalized KKM mapping with respect to T. Hence $F(x,y)$ is a properly quasimonotone mapping of Type (I) (resp., Types (II) and (III)) with respect to T.

Proposition 4.3 *Let $(Y, \{\varphi_N\})$ be a* FC*-space, X be a topological space and Z be a nonempty set. Let $T : Y \to 2^X$, $F : X \times Y \to 2^Z$, and $C : X \to 2^Z$ be set-valued mappings. Suppose that*

(1) *for all $y \in Y$ and $x \in T(y)$, $F(x,y) \subset C(x)$ (resp., $F(x,y) \not\subset C(x)$, $F(x,y) \cap C(x) \neq \emptyset$),*
(2) *for each $x \in X$, the set $\{y \in Y : F(x,y) \not\subset C(x)\}$ (resp., $\{y \in Y : F(x,y) \subset C(x)\}$, $\{y \in Y : F(x,y) \cap C(x) = \emptyset\}$) is a* FC*-subspace of Y.*

Then $F(x,y)$ is a properly quasimonotone mapping of Type (I) (*resp., Types* (II) *and* (III)) *with respect to T.*

Proof Define a set-valued mapping $F^* : Y \to 2^X$ by

$$F^*(y) = \{x \in X : F(x,y) \subset C(x)\}$$
$$(\text{resp., } F^*(y) = \{x \in X : F(x,y) \not\subset C(x)\},\ F^*(y) = \{x \in X : F(x,y) \cap C(x) \neq \emptyset\}).$$

Then for each $x \in X$, we have

$$Y \setminus (F^*)^{-1}(x) = \{y \in Y : F(x,y) \not\subset C(x)\}$$
$$(\text{resp., } Y \setminus (F^*)^{-1}(x) = \{y \in Y : F(x,y) \subset C(x)\},\ Y \setminus (F^*)^{-1}(x) = \{y \in Y : F(x,y) \bigcap C(x) = \emptyset\}).$$

By (2), for each $x \in X$, we have $Y \backslash (F^*)^{-1}(x) = \{y \in Y : F(x,y) \not\subset C(x)\}$ (resp., $Y \backslash (F^*)^{-1}(x) = \{y \in Y : F(x,y) \subset C(x)\}$, $Y \backslash (F^*)^{-1}(x) = \{y \in Y : F(x,y) \cap C(x) = \emptyset\}$) is a FC-subspace of Y. By (2), we have that for each $x \in X$, $Y \backslash (F^*)^{-1}(x) \subset Y \backslash T^{-1}(x)$. Hence for each $x \in X$, $Y \backslash T^{-1}(x)$ is a FC-subspace of Y relative to $\{y \in Y : F(x,y) \not\subset C(x)\}$ (resp., $\{y \in Y : F(x,y) \subset C(x)\}$, $\{y \in Y : F(x,y) \cap C(x) = \emptyset\}$). By Proposition 4.2, the conclusion of Proposition 4.3 holds.

Remark 4.3 Propositions 4.2 and 4.3 generalize Proposition 4.1 of Lin and Chen [22] from topological vector spaces to FC-spaces

Proposition 4.4 *Let X and Y be topological spaces and Z be a nonempty sets. Let $F: X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings. $F(x,y)$ is a C-transfer compactly continuous mapping of Type* (I) (*resp., Types* (II) *and* (III)) *in first argument if and only if the mapping $F^*: Y \to 2^X$ defined by*

$$F^*(y) = \{x \in X : F(x,y) \subset C(x)\}$$
$$(\text{resp., } F^*(y) = \{x \in X : F(x,y) \not\subset C(x)\},\ \ F^*(y) = \{x \in X : F(x,y) \bigcap C(x) \neq \emptyset\})$$

is a transfer compactly closed-valued mapping.

Proof Define a set-valued mapping $H: Y \to 2^X$ by

$$H(y) = X \backslash F^*(y), \quad \forall\, y \in Y.$$

It is enough to show that $F(x,y)$ is a C-transfer compactly continuous mapping of Type (I) (resp., Types (II) and (III)) in first argument if and only if H is a transfer compactly open-valued mapping in first argument by Lemma 2.2 of Ding [13]. Suppose that $F(x,y)$ is a C-transfer compactly continuous mapping of Type (I) (resp., Types (II) and (III)) in first argument. Then for any compact subset K of X and any $x \in K$, if $x \in H(y) \cap K$, we have $\{y \in Y : F(x,y) \not\subset C(x)\} \neq \emptyset$ (resp., $\{y \in Y : F(x,y) \subset C(x)\} \neq \emptyset$, $\{y \in Y : F(x,y) \cap C(x) = \emptyset\} \neq \emptyset$). By definition 4.3, there exist a relatively open neighborhood $N(x)$ of x in K and a point $y' \in Y$ such that $F(z,y') \not\subset C(z)$ (resp., $F(z,y') \subset C(z)$, $F(z,y') \cap C(z) = \emptyset$) for all $z \in N(x)$. It follows that

$$N(x) \subset \text{int}_K(\{z \in K : F(z,y') \not\subset C(z)\}) = \text{int}_K(H(y') \cap K)$$
$$(\text{resp., } N(x) \subset \text{int}_K(\{z \in K : F(z,y') \subset C(z)\}) = \text{int}_K(H(y') \cap K),$$
$$N(x) \subset \text{int}_K(\{z \in K : F(z,y') \cap C(z) = \emptyset\}) = \text{int}_K(H(y') \cap K)).$$

Hence H is a transfer compactly open-valued mapping. Now suppose that H is a transfer compactly open-valued mapping. Then, for any compact subset K of X and any $x \in K$, if $\{y \in Y : F(x,y) \not\subset C(x)\} \neq \emptyset$ (resp., $\{y \in Y : F(x,y) \subset C(x)\} \neq \emptyset$, $\{y \in Y : F(x,y) \cap C(x) = \emptyset\} \neq \emptyset$), then there exists $y \in Y$ such that $x \in H(y) \cap K$. Since H a transfer compactly open-valued mapping, there exists $y' \in Y$ such that $x \in \text{int}_K(H(y') \cap K)$. It follows that there exists a relatively open neighborhood $N(x)$ of x in K such that $N(x) \subset \text{int}_K(H(y') \cap K)$. Hence we have $F(z,y') \not\subset C(z)$ (resp., $F(z,y') \subset C(z)$, $F(z,y') \cap C(z) = \emptyset$) for all $z \in N(x)$. This show that $F(x,y)$ is a C-transfer compactly continuous mapping of Type (I) (resp., Type (II), Type (III)) in first argument.

Remark 4.4 Proposition 4.4 generalizes Lemma 2.2 of Ding and Park [15].

Proposition 4.5 *Let X, Y and Z be topological spaces. Let $F : X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings such that*

(1) *C has closed* (*resp., open*) *graph,*
(2) *for each* $y \in Y$, $F(\cdot, y)$ *is lower semicontinuous on each compact subset of* X.

Then the mapping $F^*: Y \to 2^X$ defined by $F^*(y) = \{x \in X: F(x,y) \subset C(x)\}$ (resp., $F^*(y) = \{x \in X: F(x,y) \cap C(x) = \emptyset\}$) has compactly closed values.

Proof For any $y \in Y$ and any compact subset K of X, if $x_0 \in \mathrm{cl}_K(K \cap F^*(y))$, then there exists a net $\{x_\alpha\}$ in $K \cap F^*(y)$ such that $x_\alpha \to x_0$. Hence we have $x_0 \in K$ and $F(x_\alpha, y) \subset C(x_\alpha)$ (resp., $F(x_\alpha, y) \cap C(x_\alpha) = \emptyset$) for all α. Let $z \in F(x_0, y)$. By (2) and Lemma 2.1, there exists a net $\{z_\alpha\} \subset F(x_\alpha, y)$ such that $z_\alpha \to z$. Thus $z_\alpha \in C(x_\alpha)$ (resp., $z_\alpha \in Z \setminus C(x_\alpha)$ for all α. By (1), we have $z \in C(x_0)$ (resp., $z \in Z \setminus C(x_0)$). It follows that $F(x_0, y) \subset C(x_0)$ (resp., $F(x_0, y) \cap C(x_0) = \emptyset$) and hence $x_0 \in K \cap F^*(y)$. Therefore the mapping $F^*: X \to 2^Y$ defined by $F^*(y) = \{x \in X: F(x,y) \subset C(x)\}$ (resp., $F^*(y) = \{x \in X: F(x,y) \cap C(x) = \emptyset\}$) has compactly closed values.

Proposition 4.6 *Let X, Y, and Z be topological spaces. Let* $F: X \times Y \to 2^Z$ *and* $C: X \to 2^Z$ *be set-valued mappings such that*

(1) *C has open* (*resp., closed*) *graph in* $X \times Z$,
(2) *for each* $y \in Y$, $F(\cdot, y)$ *is upper semicontinuous on each compact subset of X with nonempty compactly closed values.*

Then the mapping $F^*: Y \to 2^X$ defined by $F^*(y) = \{x \in X: F(x,y) \not\subset C(x)\}$ (resp., $F^*(y) = \{x \in X: F(x,y) \cap C(x) \neq \emptyset\}$) has compactly closed values.

Proof For any $y \in Y$ and any compact subset K of X, if $x_0 \in \mathrm{cl}_K(K \cap F^*(y))$, then there exists a net $\{x_\alpha\}$ in $K \cap F^*(y)$ such that $x_\alpha \to x_0$. Hence we have $x_0 \in K$ and $F(x_\alpha, y) \not\subset C(x_\alpha)$ (resp., $F(x_\alpha, y) \cap C(x_\alpha) \neq \emptyset$) for all α. Hence there exists $z_\alpha \in F(x_\alpha, y)$ such that $z_\alpha \in Z \setminus C(x_\alpha)$ (resp., $z_\alpha \in C(x_\alpha)$) for all α. By condition (2) and Proposition 3.1.11 of Aubin ana Ekeland [3], the set $\bigcup_{x \in K} F(x,y)$ is compact in Z. Since $\{z_\alpha\} \subset \bigcup_{x \in K} F(x,y)$, without loss of generality we can assume that $z_\alpha \to z$. By the upper semicontinuity of $F(\cdot, y)$, we have $z \in F(x_0, y)$. Since C has open (resp., Closed) graph in $X \times Z$, we have $z \notin C(x_0)$ (resp., $z \in C(x_0)$). Hence $x_0 \in K \cap F^*(y)$ and so $F^*(y)$ has compactly closed values.

Remark 4.5 Proposition 4.6 generalizes Lemma 2.3 of Ding and Park [15].

Theorem 4.1 *Let* $(Y, \{\varphi_N\})$ *be a* FC-*space,* X *be a topological space and* Z *be a nonempty set. Let* $T \in \mathrm{KKM}(Y, X)$, $F, G, M: X \times Y \to 2^Z$ *and* $C: X \to 2^Z$ *be set-valued mappings such that*

(1) $F(x,y)$ *is a C-transfer compactly continuous mapping of Type* (I) (*resp., Types* (II) *and* (III)) *in first argument,*
(2) *for each* $y \in Y$ *and* $x \in T(y)$, $M(x,y) \subset C(x)$ (*resp.,* $M(x,y) \not\subset C(x)$, $M(x,y) \cap C(x) \neq \emptyset$),
(3) *for each* $(x,y) \in X \times Y$, $F(x,y) \subset G(x,y)$, (*resp.,* $G(x,y) \not\subset C(x)$ *implies* $F(x,y) \not\subset C(x)$, $G(x,y) \cap C(x) \neq \emptyset$ *implies* $F(x,y) \cap C(x) \neq \emptyset$),
(4) $G(x,y)$ *is a C-diagonally quasi-convex mapping of Type* (I) (*resp., Types* (II) *and* (III)) *with respect to the mapping* $M^*: Y \to 2^X$ *defined by* $M^*(y) = \{x \in X: M(x,y) \subset C(x)\}$ (*resp.,* $M^*(y) = \{x \in X: M(x,y) \not\subset C(x)\}$, $M^*(y) = \{x \in X: M(x,y) \cap C(x) \neq \emptyset\}$) *in second argument,*
(5) *for each compact subset D of Y,* $\overline{T(D)}$ *is compact in X,*

(6) *there exists compact subset K of X such that for each $N \in \langle Y \rangle$, there exists a compact* FC-*subspace L_N of Y containing N satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x,y) \subset C(x)\} \right) \subset K$$

$$(\text{resp., } T(L_N) \cap \ (\cap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x,y) \not\subset C(x)\}) \subset K,$$

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x,y) \cap C(x) \neq \emptyset\} \right) \subset K).$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \subset C(\hat{x}) \ \ (\text{resp., } F(\hat{x}, y) \not\subset C(\hat{x}), \ \ F(\hat{x}, y) \cap C(\hat{x}) \neq \emptyset), \quad \forall\, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(I) (resp., GVEP(II), GVEP(III)) .

Proof Define set-valued mappings $F^*, G^*, M^* : Y \to 2^X$ by

$$F^*(y) = \{x \in X : F(x,y) \subset C(x)\}$$

$$(\text{resp., } F^*(y) = \{x \in X : F(x,y) \not\subset C(x)\}, \ \ F^*(y) = \{x \in X : F(x,y) \cap C(x) \neq \emptyset\}),$$

$$G^*(y) = \{x \in X : G(x,y) \subset C(x)\}$$

$$(\text{resp., } G^*(y) = \{x \in X : G(x,y) \not\subset C(x)\}, \ \ G^*(y) = \{x \in X : G(x,y) \cap C(x) \neq \emptyset\}), \ \forall\, y \in Y.$$

Then, by (1) and Proposition 4.4, F^* is transfer compactly closed-valued. By (2) and (3), we have that for each $y \in Y$, $T(y) \subset M^*(y)$, and $G^*(y) \subset F^*(y)$. By (4), Proposition 4.1 and Definition 4.1, G^* is a generalized *KKM* mapping with respect to M^*. The conditions (5) and (6) imply that the conditions (4) and (5) of Theorem 3.2 hold. By Theorem 3.2, $\cap_{y \in Y} F^*(y) \neq \emptyset$. Taking $\hat{x} \in \cap_{y \in Y} F^*(y)$, we obtain

$$F(\hat{x}, y) \subset C(\hat{x}) \ \ (\text{resp., } F(\hat{x}, y) \not\subset C(\hat{x}), \ \ F(\hat{x}, y) \bigcap C(\hat{x}) \neq \emptyset), \quad \forall\, y \in Y.$$

Corollary 4.1 *Let $(Y, \{\varphi_N\})$ be a* FC-*space, X be a topological space and Z be a non-empty set. Let $T \in$* KKM(Y, X)*, $F : X \times Y \to 2^Z$ and $C : X \to 2^Z$ be set-valued mappings such that*

(1) *$F(x,y)$ is a C-transfer compactly continuous mapping of Type* (I) (*resp., Types* (II) *and* (III)) *in first argument,*
(2) *for each $y \in Y$ and $x \in T(y)$, $F(x,y) \subset C(x)$ (resp., $F(x,y) \not\subset C(x)$, $F(x,y) \cap C(x) \neq \emptyset$),*
(3) *$F(x,y)$ is a C-diagonally quasi-convex mapping of Type* (I) (*resp., Types* (II) *and Type* (III)) *with respect to the mapping T in second argument,*
(4) *for each compact subset D of Y, $\overline{T(D)}$ is compact in X,*
(5) *there exists compact subset K of X such that for each $N \in \langle Y \rangle$, there exists a compact* FC-*subspace L_N of Y containing N satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x,y) \subset C(x)\} \right) \subset K$$

$$(\text{resp., } T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x,y) \not\subset C(x)\} \right) \subset K,$$

$$T(L_N) \cap \ (\cap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x,y) \cap C(x) \neq \emptyset\}) \subset K).$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \subset C(\hat{x}) \text{ (resp., } F(\hat{x}, y) \not\subset C(\hat{x}), \ F(\hat{x}, y) \bigcap C(\hat{x}) \neq \emptyset), \ \forall y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(I) (resp., GVEP(II), GVEP(III)) .

Proof The conclusion of Corollary 4.1 holds from Theorem 4.1 with $F = G = M$.

Corollary 4.2 *Let* $(Y, \{\varphi_N\})$ *be a* FC-*space,* X *and* Z *be topological spaces. Let* $T \in \mathrm{KKM}(Y, X)$*,* $F: X \times Y \to 2^Z$ *and* $C: X \to 2^Z$ *be set-valued mappings such that*

(1) *for each* $y \in Y$*,* $F(\cdot, y)$ *is lower semicontinuous in each compact subset of* X *and* C *has closed graph,*
(2) *for each* $y \in Y$ *and* $x \in T(y)$*,* $F(x, y) \subset C(x)$*,*
(3) $F(x, y)$ *is a* C*-diagonally quasi-convex mapping of Type* (I) *with respect to the mapping* T *in second argument,*
(4) *for each compact subset* D *of* Y*,* $\overline{T(D)}$ *is compact in* X*,*
(5) *there exists compact subset* K *of* X *such that for each* $N \in \langle Y \rangle$*, there exists a compact* FC-*subspace* L_N *of* Y *containing* N *satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x, y) \subset C(x)\} \right) \subset K.$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \subset C(\hat{x}) \ \forall y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(I).

Proof Define a set-valued mapping $F^*: Y \to 2^X$ by $F^*(y) = \{x \in X : F(x, y) \subset C(x)\}$ for each $y \in Y$. Then, by (1) and Proposition 4.5, F^* has compactly closed values and hence F^* is a transfer compactly closed-valued mapping. By Proposition 4.4, $F(x, y)$ is a C-transfer compactly continuous mapping of Type (I) in first argument. The condition (1) of Corollary 4.1 is satisfied. The conclusion of Corollary 4.2 hold from Corollary 4.1.

As a consequence of Corollary 4.2 with $X = Y$ and T being the identity mapping, we have the following result.

Corollary 4.3 *Let* $(X, \{\varphi_N\})$ *be a* FC-*space, and* Z *be a topological space. Let* $F : X \times X \to 2^Z$ *and* $C: X \to 2^Z$ *be set-valued mappings such that*

(1) *for each* $y \in X$*,* $F(\cdot, y)$ *is lower semicontinuous in each compact subset of* X *and* C *has closed graph,*
(2) $F(x, y)$ *is a* C*-diagonally quasi-convex mapping of Type* (I) *in second argument,*
(3) *there exists compact subset* K *of* X *such that for each* $N \in < X >$*, there exists a compact* FC-*subspace* L_N *of* X *containing* N *satisfying*

$$L_N \backslash K \subset \bigcup_{y \in L_N} \mathrm{cint}\{x \in X : F(x, y) \subset C(x)\}.$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \subset C(\hat{x}) \ \ \forall y \in X.$$

Corollary 4.4 *Let $(Y, \{\varphi_N\})$ be a* FC-*space, X and Z be topological spaces. Let $T \in$* KKM(Y, X)*, $F: X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings such that*

(1) *for each $y \in Y$, $F(\cdot, y)$ is upper semicontinuous in each compact subset of X with nonempty compact values and C has open (resp., closed) graph in $X \times Z$,*
(2) *for each $y \in Y$ and $x \in T(y)$, $F(x, y) \not\subset C(x)$, (resp., $F(x, y) \cap C(x) \neq \emptyset$),*
(3) *$F(x, y)$ is a C-diagonally quasi-convex mapping of Type* (II) (*resp., Type* (III)) *with respect to the mapping T in second argument,*
(4) *for each compact subset D of Y, $\overline{T(D)}$ is compact in X,*
(5) *there exists compact subset K of X such that for each $N \in \langle Y \rangle$, there exists a compact* FC-*subspace L_N of Y containing N satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X: F(x, y) \not\subset C(x)\} \right) \subset K$$

$$(\text{resp., } T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X: F(x, y) \cap C(x) \neq \emptyset\} \right) \subset K).$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \not\subset C(\hat{x}) \ \ (\text{resp., } F(\hat{x}, y) \bigcap C(\hat{x}) \neq \emptyset), \quad \forall\, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(II) (resp. GVEP(III)) .

Proof Define a set-valued mapping $F^*: Y \to 2^X$ by $F^*(y) = \{x \in X: F(x, y) \not\subset C(x)\}$ (resp., $F^*(y) = \{x \in X: F(x, y) \cap C(x) \neq \emptyset\}$) for each $y \in Y$. Then, by (1) and Proposition 4.6, F^* has compactly closed values and hence F^* is a transfer compactly closed-valued mapping. By Proposition 4.4, $F(x, y)$ is a C-transfer compactly continuous mapping of Type (II) (resp., Type (III)) in first argument. The condition (i) of Corollary 4.1 is satisfied. The conclusion of Corollary 4.3 hold from Corollary 4.1.

As a consequence of Corollary 4.4 with $X = Y$ and T being the identity mapping, we have the following result.

Corollary 4.5 *Let $(X, \{\varphi_N\})$ be a* FC-*space, and Z be a topological space. Let F : $X \times X \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings such that*

(1) *for each $y \in X$, $F(\cdot, y)$ is upper semicontinuous in each compact subset of X with nonempty compact values and C has open (resp., closed) graph in $X \times Z$,*
(2) *$F(x, y)$ is a C-diagonally quasi-convex mapping of Type* (II) (*resp., Type* (III)) *in second argument,*
(3) *there exists compact subset K of X such that for each $N \in < X >$, there exists a compact* FC-*subspace L_N of X containing N satisfying*

$$L_N \backslash K \subset \textstyle\bigcup_{y \in L_N} \mathrm{cint}\{x \in X: F(x, y) \not\subset C(x)\}$$
$$(\text{resp., } L_N) \backslash K \subset \textstyle\bigcup_{y \in L_N} \mathrm{cint}\{x \in X: F(x, y) \cap C(x) \neq \emptyset\}).$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \not\subset C(\hat{x}) \ \ (\text{resp., } F(\hat{x}, y) \bigcap C(\hat{x}) \neq \emptyset),\ \forall\, y \in X.$$

Remark 4.6 Theorem 4.1 and Corollaries 4.1–4.5 improves and generalizes Theorems 4.1–4.4, 4.6, 4.9 and 4.10 of Lin and Chen [22] from topological vector spaces to FC-spaces without any convexity structure under much weak assumptions.

Corollary 4.6 *Let $(Y, \{\varphi_N\})$ be a* FC*-space, X be a topological space and Z be a non-empty set. Let $T \in \mathrm{KKM}(Y, X)$ be a compact mapping, $F: X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings such that*

(1) *$F(x, y)$ is a C-transfer compactly continuous mapping of Type* (I) (*resp., Types* (II) *and Type* (III)) *in frist argument,*
(2) *for each $y \in Y$ and $x \in T(y)$, $F(x, y) \subset C(x)$ (resp., $F(x, y) \not\subset C(x)$, $F(x, y) \cap C(x) \neq \emptyset$),*
(3) *$F(x, y)$ is a C-diagonally quasi-convex mapping with respect to T in second argument, Then there exists $\hat{x} \in X$ such that*

$$F(\hat{x}, y) \subset C(\hat{x}) \ \ (\text{resp.}, F(\hat{x}, y) \not\subset C(\hat{x}), \ \ F(\hat{x}, y) \cap C(\hat{x}) \neq \emptyset), \quad \forall\, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(I) (resp., GVEP(II), GVEP(III)) .

Proof Since T is a compact mapping, $\overline{T(Y)}$ is compact in X. The conditions (4) and (5) of Corollary 4.1 are satisfied trivially. The conclusion of Corollary 4.6 holds from Corollary 4.1.

Theorem 4.2 *Let $(Y, \{\varphi_N\})$ be a* FC*-space, X be a topological space and Z be a non-empty set. Let $T \in \mathrm{KKM}(Y, X)$, $F, G, M: X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings such that*

(1) *$F(x, y)$ is a C-transfer compactly continuous mapping of Type* (I) (*resp., Types* (II) *and Type* (III)) *in first argument,*
(2) *for each $y \in Y$ and $x \in T(y)$, $M(x, y) \subset C(x)$ (resp., $M(x, y) \not\subset C(x)$, $M(x, y) \cap C(x) \neq \emptyset$),*
(3) *for each $(x, y) \in X \times Y$, $F(x, y) \subset G(x, y)$, (resp., $G(x, y) \not\subset C(x)$ implies $F(x, y) \not\subset C(x)$, $G(x, y) \cap C(x) \neq \emptyset$ implies $F(x, y) \cap C(x) \neq \emptyset$),*
(4) *for each $x \in X$, $\{y \in Y: M(x, y) \not\subset C(x)\}$ (resp., $\{y \in Y: M(x, y) \subset C(x)\}$, $\{y \in Y: M(x, y) \cap C(x) = \emptyset\}$) is a* FC*-subspace of Y relative to $\{y \in Y: G(x, y) \not\subset C(x)\}$ (resp., $\{y \in Y: G(x, y) \subset C(x)\}$, $\{y \in Y: G(x, y) \cap C(x) = \emptyset\}$),*
(5) *for each compact subset D of Y, $\overline{T(D)}$ is compact in X,*
(6) *there exists compact subset K of X such that for each $N \in \langle Y \rangle$, there exists a compact* FC*-subspace L_N of Y containing N satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X: F(x, y) \subset C(x)\} \right) \subset K$$

$$(\text{resp.}, T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X: F(x, y) \not\subset C(x)\} \right) \subset K,$$

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X: F(x, y) \cap C(x) \neq \emptyset\} \right) \subset K).$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \subset C(\hat{x}) \ \ (\text{resp.}, F(\hat{x}, y) \not\subset C(\hat{x}), \ \ F(\hat{x}, y) \bigcap C(\hat{x}) \neq \emptyset), \quad \forall\, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(I) (resp., GVEP(II), GVEP(III)) .

Proof Define set-valued mappings $F^*, G^*, M^* \colon Y \to 2^X$ by

$$F^*(y) = \{x \in X \colon F(x,y) \subset C(x)\}$$
$$(\text{resp., } F^*(y) = \{x \in X \colon F(x,y) \not\subset C(x)\},\ \ F^*(y) = \{x \in X \colon F(x,y) \cap C(x) \neq \emptyset\}),$$
$$G^*(y) = \{x \in X \colon G(x,y) \subset C(x)\}$$
$$(\text{resp., } G^*(y) = \{x \in X \colon G(x,y) \not\subset C(x)\},\ \ G^*(y) = \{x \in X \colon G(x,y) \cap C(x) \neq \emptyset\}), \text{and}$$
$$M^*(y) = \{x \in X \colon M(x,y) \subset C(x)\}$$
$$(\text{resp., } M^*(y) = \{x \in X \colon M(x,y) \not\subset C(x)\},\ \ M^*(y) = \{x \in X \colon M(x,y) \cap C(x) \neq \emptyset\}),\ \forall\, y \in Y.$$

By the conditions (1)–(3), (5),(6) and using same argument as in the proof of Theorem 4.1, the conditions (1),(2), (4) and (5) of Theorem 3.2 are satisfied. Note that for each $x \in X$, $Y \setminus (M^*)^{-1}(x) = \{y \in Y : M(x,y) \not\subset C(x)\}$ (resp., $Y \setminus (M^*)^{-1}(x) = \{y \in Y : M(x,y) \subset C(x)\}$, $Y \setminus (M^*)^{-1}(x) = \{y \in Y : M(x,y) \cap C(x) = \emptyset\}$) and $Y \setminus (G^*)^{-1}(x) = \{y \in Y \colon G(x,y) \not\subset C(x)\}$ (resp., $Y \setminus (G^*)^{-1}(x) = \{y \in Y \colon G(x,y) \subset C(x)\}$, $Y \setminus (G^*)^{-1}(x) = \{y \in Y \colon G(x,y) \cap C(x) = \emptyset\}$). By (4), Proposition 4.2 and Definition 4.1, G^* is a generalized KKM mapping with respect to M^*. the condition (3) of Theorem 3.2 is satisfied. By Theorem 3.2, $\cap_{y \in Y} F^*(y) \neq \emptyset$. Taking $\hat{x} \in \cap_{y \in Y} F^*(y)$, we obtain

$$F(\hat{x},y) \subset C(\hat{x})\ \ (\text{resp., } F(\hat{x},y) \not\subset C(\hat{x}),\ \ F(\hat{x},y) \cap C(\hat{x}) \neq \emptyset), \quad \forall\, y \in Y.$$

As consequence of Theorem 4.2, we have the following results.

Corollary 4.7 *Let $(Y, \{\varphi_N\})$ be a* FC*-space, X be a topological space and Z be a nonempty set. Let $T \in$* KKM(Y,X)*, $F : X \times Y \to 2^Z$ and $C : X \to 2^Z$ be set-valued mappings such that*

(1) *$F(x,y)$ is a C-transfer compactly continuous mapping of Type (I) (resp., Types* (II) *and* (III)) *in first argument,*
(2) *for each $y \in Y$ and $x \in T(y)$, $F(x,y) \subset C(x)$ (resp., $F(x,y) \not\subset C(x)$, $F(x,y) \cap C(x) \neq \emptyset$),*
(3) *for each $x \in X$, $\{y \in Y : F(x,y) \not\subset C(x)\}$ (resp., $\{y \in Y : F(x,y) \subset C(x)\}$, $\{y \in Y \colon F(x,y) \cap (x) = \emptyset\}$) is a* FC*-subspace of Y,*
(4) *for each compact subset D of Y, $\overline{T(D)}$ is compact in X,*
(5) *there exists compact subset K of X such that for each $N \in \langle Y \rangle$, there exists a compact* FC*-subspace L_N of Y containing N satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \text{ccl}\{x \in X \colon F(x,y) \subset C(x)\} \right) \subset K$$
$$(\text{resp., } T(L_N) \bigcap \left(\bigcap_{y \in L_N} \text{ccl}\{x \in X \colon F(x,y) \not\subset C(x)\} \right) \subset K,$$
$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \text{ccl}\{x \in X \colon F(x,y) \cap C(x) \neq \emptyset\} \right) \subset K).$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x},y) \subset C(\hat{x})\ \ (\text{resp., } F(\hat{x},y) \not\subset C(\hat{x}),\ \ F(\hat{x},y) \cap C(\hat{x}) \neq \emptyset), \quad \forall\, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(I) (resp., GVEP(II), GVEP(III)) .

Corollary 4.8 *Let $(Y, \{\varphi_N\})$ be a* FC*-space, X be a topological space and Z be a nonempty set. Let $T \in \mathrm{KKM}(Y, X)$ be a compact mapping, $F: X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings such that*

(1) *$F(x, y)$ is a C-transfer compactly continuous mapping of Type (I) (resp., Types* (II) *and* (III)) *in first argument,*
(2) *for each $y \in Y$ and $x \in T(y)$, $F(x, y) \subset C(x)$ (resp., $F(x, y) \not\subset C(x)$, $F(x, y) \cap C(x) \neq \emptyset$),*
(3) *for each $x \in X$, $\{y \in Y : F(x, y) \not\subset C(x)\}$ (resp., $\{y \in Y : F(x, y) \subset C(x)\}$, $\{y \in Y : F(x, y) \cap C(x) = \emptyset\}$) is a* FC*-subspace of Y, Then there exists $\hat{x} \in X$ such that*

$$F(\hat{x}, y) \subset C(\hat{x}) \ \ (\text{resp.}, F(\hat{x}, y) \not\subset C(\hat{x}), \ \ F(\hat{x}, y) \cap C(\hat{x}) \neq \emptyset), \quad \forall\, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(I) (resp., GVEP(II), GVEP(III)) .

Theorem 4.3 *Let $(Y, \{\varphi_N\})$ be a* FC*-space, X and Z be topological spaces. Let $T \in \mathrm{KKM}(Y, X)$, $F, G, M: X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings such that*

(1) *for each $y \in Y$, $F(\cdot, y)$ is lower semicontinuous in each compact subset of X and C has closed graph,*
(2) *for each $y \in Y$ and $x \in T(y)$, $M(x, y) \subset C(x)$ (resp., $M(x, y) \cap C(x) \neq \emptyset$),*
(3) *for all $(x, y) \in X \times Y$, $F(x, y) \subset G(x, y)$ implies $F(x, y) \cap C(x) \neq \emptyset$),*
(4) *for all $x \in X$, $\{y \in Y : M(x, y) \not\subset C(x)\}$ is a* FC*-subspace of Y relative to $\{y \in Y : G(x, y) \not\subset C(x)\}$,*
(5) *for each compact subset D of Y, $\overline{T(D)}$ is compact in X,*
(6) *there exists compact subset K of X such that for each $N \in \langle Y \rangle$, there exists a compact* FC*-subspace L_N of Y containing N satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x, y) \subset C(x)\} \right) \subset K$$

such that

$$F(\hat{x}, y) \subset C(\hat{x}) \ \ \forall\, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(I).

Proof Define a set-valued mapping $F^*: Y \to 2^X$ by $F^*(y) = \{x \in X : F(x, y) \subset C(x)\}$ for each $y \in Y$. Then, by (1) and Proposition 4.5, F^* has compactly closed values and hence F^* is a transfer compactly closed-valued mapping. By Proposition 4.4, $F(x, y)$ is a C-transfer compactly continuous mapping of Type (I) in first argument. The condition (1) of Theorem 4.2 is satisfied. Therefore the conclusion of Theorem 4.3 holds from Theorem 4.2.

As consequence of Theorem 4.2, we have the following results.

Corollary 4.9 *Let $(Y, \{\varphi_N\})$ be a* FC*-space, X and Z be topological spaces. Let $T \in \mathrm{KKM}(Y, X)$, $F: X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings such that*

(1) *for each $y \in Y$, $F(\cdot, y)$ is lower semicontinuous in each compact subset of X and C has closed graph,*
(2) *for each $y \in Y$ and $x \in T(y)$, $F(x, y) \subset C(x)$,*
(3) *for each $x \in X$, $\{y \in Y : F(x, y) \not\subset C(x)\}$ is a* FC*-subspace of Y,*

(4) *for each compact subset D of Y, $\overline{T(D)}$ is compact in X,*
(5) *there exists compact subset K of X such that for each $N \in \langle Y \rangle$, there exists a compact* FC*-subspace L_N of Y containing N satisfying*

$$T(L_N) \cap \; (\cap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x,y) \subset C(x)\}) \subset K.$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \subset C(\hat{x}) \;\; \forall \, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(I).

Remark 4.7 Theorem 4.3 and Corollary 4.9 generalizes Theorems 4.1, 4.3 of Lin and Chen [22] from topological vector spaces to FC-spaces without any convexity structure.

Corollary 4.10 *In Corollary* 4.9, *if $T \in \mathrm{KKM}(Y, X)$ is replaced by that T being an upper semicontinuous set-valued mapping with nonempty compact acyclic values and assume that the conditions* (1)–(3) *and* (5) *hold.*
Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \subset C(\hat{x}) \;\; (\text{resp., } \; F(\hat{x}, y) \bigcap C(\hat{x}) \neq \emptyset), \quad \forall \, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(I) (resp. GVEP(III)) .

Proof Since T is upper semicontinuous with nonempty compact acyclic values, $T \in V(Y, X) \subset \mathrm{KKM}(Y, X)$. By Proposition 3.1.11 of Aubin and Ekeland [2], for any compact subset D of Y, $T(D) = \overline{T(D)}$ is compact in X. Hence the conclusion of Corollary 4.10 holds from Corollary 4.9.

Remark 4.8 Corollary 4.10 improves and generalizes Corollary 4.1 of Lin and Chen [22] to FC-spaces under weaker assumptions.

Theorem 4.4 *Let $(Y, \{\varphi_N\})$ be a* FC*-space, X and Z be topological spaces. Let $T \in$* KKM(Y, X)*, $F, G, M : X \times Y \to 2^Z$ and $C : X \to 2^Z$ be set-valued mappings such that*

(1) *for each $y \in Y$, $F(\cdot, y)$ is upper semicontinuous in each compact subset of X with nonempty compact values and C has open (resp., closed) graph in $X \times Z$,*
(2) *for each $y \in Y$ and $x \in T(y)$, $M(x, y) \not\subset C(x)$ (resp., $M(x, y) \cap C(x) \neq \emptyset$),*
(3) *for each $(x, y) \in X \times Y$, $G(x, y) \not\subset C(x)$ implies $F(x, y) \not\subset C(x)$, (resp., $G(x, y) \cap C(x) \neq \emptyset$ implies $F(x, y) \cap C(x) \neq \emptyset$),*
(4) *for each $x \in X$, $\{y \in Y : M(x, y) \subset C(x)\}$ (resp., $\{y \in Y : M(x, y) \cap C(x) = \emptyset\}$) is a* FC*-subspace of Y relative to $\{y \in Y : G(x, y) \subset C(x)\}$ (resp., $\{y \in Y : G(x, y) \cap C(x) = \emptyset\}$),*
(5) *for each compact subset D of Y, $\overline{T(D)}$ is compact in X,*
(6) *there exists compact subset K of X such that for each $N \in \langle Y \rangle$, there exists a compact* FC*-subspace L_N of Y containing N satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x, y) \not\subset C(x)\} \right) \subset K$$

$$(\text{resp., } T(L_N) \bigcap \left(\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X : F(x, y) \cap C(x) \neq \emptyset\} \right) \subset K).$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \not\subset C(\hat{x}) \ \ (\text{resp., } F(\hat{x}, y) \bigcap C(\hat{x}) \neq \emptyset), \quad \forall\, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(II) (resp. GVEP(III)) .

Proof Define a set-valued mapping $F^*: Y \to 2^X$ by $F^*(y) = \{x \in X: F(x, y) \not\subset C(x)\}$ (resp., $F^*(y) = \{x \in X : F(x, y) \cap C(x) \neq \emptyset\}$) for each $y \in Y$. Then, by (1) and Proposition 4.6, F^* has compactly closed values and hence F^* is a transfer compactly closed-valued mapping. By Proposition 4.4, $F(x, y)$ is a C-transfer compactly continuous mapping of Type (II) (resp., Type (III)) in first argument. The condition (2) of Theorem 4.2 is satisfied. Therefore the conclusion of Theorem 4.4 holds from Theorem 4.2.

Corollary 4.11 *Let $(Y, \{\varphi_N\})$ be a* FC*-space, X and Z be topological spaces. Let $T \in$* KKM(Y, X)*, $F: X \times Y \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings such that*

(1) *for each $y \in Y$, $F(\cdot, y)$ is upper semicontinuous in each compact subset of X with nonempty compact values and C has open (resp., closed) graph in $X \times Z$,*
(2) *for each $y \in Y$ and $x \in T(y)$, $F(x, y) \not\subset C(x)$ (resp., $F(x, y) \cap C(x) \neq \emptyset$),*
(3) *for each $x \in X$, $\{y \in Y: F(x, y) \subset C(x)\}$ (resp., $\{y \in Y: F(x, y) \cap C(x) = \emptyset\}$) is a* FC*-subspace of Y,*
(4) *for each compact subset D of Y, $\overline{T(D)}$ is compact in X,*
(5) *there exists compact subset K of X such that for each $N \in \langle Y \rangle$, there exists a compact* FC*-subspace L_N of Y containing N satisfying*

$$T(L_N) \bigcap \left(\bigcap_{y \in L_N} \text{ccl}\{x \in X: F(x, y) \not\subset C(x)\} \right) \subset K$$

$$(\text{resp., } T(L_N) \bigcap \left(\bigcap_{y \in L_N} \text{ccl}\{x \in X: F(x, y) \cap C(x) \neq \emptyset\} \right) \subset K).$$

Then there exists $\hat{x} \in X$ such that

$$F(\hat{x}, y) \not\subset C(\hat{x}) \ \ (\text{resp., } F(\hat{x}, y) \bigcap C(\hat{x}) \neq \emptyset), \quad \forall\, y \in Y,$$

i.e., $\hat{x}$ is a solution of the GVEP(II) (resp. GVEP(III)) .

Proof The conclusion of Corollary 4.11 from Theorem 4.4 with $F = M = G$.

Remark 4.9 Theorem 4.4 and Corollary 11 improve and generalize Theorems 4.5, 4.7, and 4.8 of Lin and Chen [22] from topological vector spaces to FC-spaces without any convexity structure under much weak assumptions.

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